

Propagation of Errors

Let's say that you have experimentally measured two quantities that will be used to determine a third quantity. There are errors in each of the measured quantities. How does the error in each of these two measured quantities affect the value of the calculated quantity?

Assume for the sake of the discussion that we use a pendulum to indirectly measure the acceleration due to gravity according to the hypothetical relationship

$$t = 2\pi\sqrt{\frac{l}{g}}$$

The length of the pendulum, l , and the period of oscillation, t , are measured. In the process of determining the local value of g , there is an error in the measurement of the length of the pendulum Δl . Similarly, assume an error in the measured period equal to Δt . How do these errors in length and period affect the resulting value of g ? Consider the following process of analyzing error propagation. Given the above relationship that has reformulated to eliminate the square root and terms in the denominator, start by substituting $g + \Delta g$ for g , $t + \Delta t$ for t , and $l + \Delta l$ for l .

$$\begin{aligned}gt^2 &= 4\pi^2 l \\(g + \Delta g)(t + \Delta t)^2 &= 4\pi^2 (l + \Delta l) \\(g + \Delta g)(t^2 + 2t\Delta t + \Delta t^2) &= 4\pi^2 (l + \Delta l) \\gt^2 + 2gt\Delta t + g\Delta t^2 + \Delta gt^2 + 2t\Delta g\Delta t + \Delta g\Delta t^2 &= 4\pi^2 l + 4\pi^2 \Delta l\end{aligned}$$

Eliminating terms in double and triple Δ (assumed $\ll 1$) and canceling like terms (note also that $gt^2 = 4\pi^2 l$) results in

$$\Delta g = \frac{4\pi^2 \Delta l - 2gt\Delta t}{t^2}$$

Now, because the Δ terms can be considered either +/-, the maximum possible error in g will result when both error terms in the numerator are of like sign. That is, the maximum absolute error is given by the relationship

$$\Delta g_{\max} = \frac{4\pi^2 \Delta l + 2gt\Delta t}{t^2}$$

After rearranging our terms and making a substitution based on our original identity ($gt^2 = 4\pi^2 l$) we get the relative error

$$\left(\frac{\Delta g}{g}\right)_{\max} = \frac{\Delta l}{l} + \frac{2\Delta t}{t}$$

Additional pointers for making this process work well:

- 1) Most of the time it is best to simplify a relationship before making the error term substitutions. Eliminating square roots and terms in the denominator will make the task of finding the error term much simpler. For instance, consider the following range equation with error terms in s , v , h , and g . Convert the equation before beginning the required substitutions:

$$s = v \times \sqrt{\frac{2h}{g}} \quad \text{becomes} \quad s^2 g = 2v^2 h \quad \text{then}$$

$$(s + \Delta s)^2 (g + \Delta g) = 2(v + \Delta v)^2 (h + \Delta h)$$

- 2) Pay close attention to substitutions that can be made by manipulating the simplified original equation obtained from following the previous hint (e.g. $2s = gt^2$). The reason for these substitutions is to eliminate “like” terms from both sides of the equation. This allows one to find terms that commonly appear in error propagation equations. In the example immediately above, this would mean the following:

$$\frac{\Delta v}{v}, \frac{\Delta s}{s}, \frac{\Delta g}{g}, \text{ and } \frac{\Delta h}{h}$$

It should be noted that with knowledge of partial differential equations, equations for absolute and relative error can be much more easily derived (especially with complex equations).