6.9. The KAM theorem and the route to Hamiltonian chaos

Back in the early days of nonlinear dynamics, the French mathematician and physicist Henri Poincaré was working on the problem of the stability of the solar system: is the sun/planet system stable over long periods of time? The answer might seem obvious to you, since the solar system has existed for millions of years, but to those who study celestial mechanics this is an important question - and they're interested on times of the order of the age of the universe. The problem is easy enough to set up: the force of gravity acts between each pair of massive bodies (ignore the asteroid belt for simplicity), the sun and all nine planets. But it's solution had eluded scientists and mathematicians for years, in fact they couldn't even solve the problem for a sun and two planets, the so-called "**three body problem**". Without getting into too much detail let me just say that Poincaré was working on the three body problem around the turn of the 20th century using a sophisticated kind of perturbation method called canonical perturbation theory. He had reached an impasse when every attempt seemed to lead to what he called "the problem of small divisors", *i.e*. series expansions with denominators that approached zero and did not converge. As Poincaré expressed it [V. Szebehely, *Theory of Orbits - The Restricted Problem of Three Bodies*, Academic Press, New York and London, 1967]: "Difficulties in celestial Mechanics due to the existence of small divisors and the associated approximate Commensurabilities of the mean motions are connected with the actual nature of things and cannot be circumvented." In other words, he was suggesting that such systems are simply not solvable mathematically. Of course he didn't have a computer to follow orbits numerically, so he really didn't have good intuition for the kinds of behavior that can exist, but he did conjecture that it was unpredictable: "it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible..." [Poincaré, *Science and Method*, Dover Publications, 2003].

Poincaré was at an impasse. Unfortunately for the study of nonlinear dynamics (but fortunately for physics in general!) other topics took over most physicists' interests: relativity and quantum theory -- and little work was done on nonlinear classical mechanics problems until the mid-20th century. By that time more mathematical tools had been developed so researchers could once again tackle the problems first enunciated by Poincaré. For example, work on equations modeling nonlinear circuits had lead to attracting sets with bizarre mathematical properties. The work of Birkhoff and Smale had further clarified Poincare's idea that some nonlinear systems were insoluble. A major question that remained for both Hamiltonian (nondissipative) and for dissipative systems was: does any degree of nonlinearity caused nonintegrability? If we increase a nonlinear perturbation does the system suddenly go crazy? In short, how do systems go chaotic? We'll leave the dissipative side of this question to the second part of the course but I'll summarize the solution to the Hamiltonian version of this question: the KAM theorem.

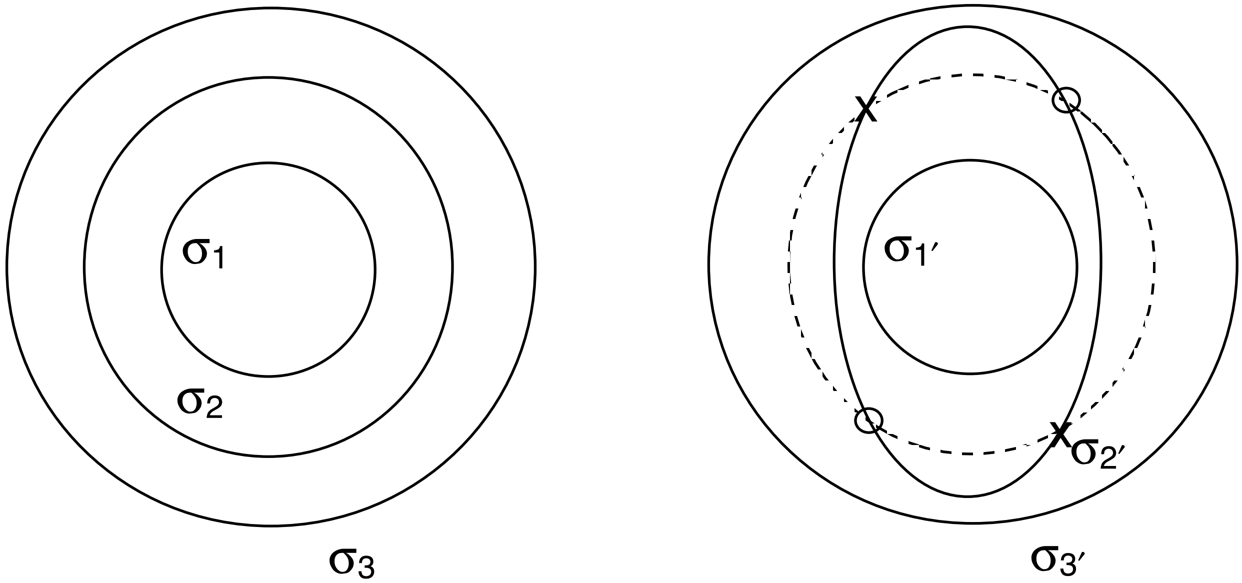
K, A, and M are Russian mathematicians with interests in mathematical physics: Kolmogorov, Arnol'd and Moser. By the mid-20th century the Soviet Union boasted some of the best mathematical physicists in the world (some argue that was because they didn't have enough money to provide their scientists with state-of-the art lab and computer facilities, but I'll let the historians and sociologists debate that one). Their theorem is couched in the language of "invariant tori" so let's digress briefly to refresh our memories on tori. Recall when we were giving examples of what the SOS plot for quasiperiodic motion looks like, we saw some sample

orbits of the double pendulum - and they looked like distorted tori. Not all orbits look like this in position-velocity space, but "motion on a torus" is the fundamental kind of quasiperiodic motion, even in higher dimensions than three. Why? Well, consider your basic torus - it's defined by two angles: one inside the "donut" tube and another around the vertical axis of the donut. In advanced mechanics, nondissipative Newtonian systems are expressed in terms of "action-angle" coordinates, consisting of "action" variables and "angle" variables, and yielding a natural visualization in terms of tori defined by the angle variables. Quasiperiodic orbits, then represent integrable motion on a torus and, if it's integrable then there is a constant, or invariant, of the motion associated with the torus that leads to the terminology "invariant tori". OK, we're now ready to state the problem that KAM solved: what happens to the invariant tori as the nonlinearity of the system increases? Here's their result , which was considered quite surprising:

For sufficiently small perturbations, almost all tori are preserved.

I suspect you have two questions: (1) what does "almost all" mean, and (2) why is this surprising? The second question has a quick (but not complete) answer: because of Poincaré's problem of small divisors, it was thought that such nonlinear perturbations would potentially destroy all tori, but KAM restored a measure of continuity to chaos. The first question takes a bit more background: consider the frequency of motion around each angular variable of a torus: as a point moves it may rotate around the "tube" at the same time it revolves around the torus axis. If you take the ratio of these frequencies you get a quantity called the **winding number**, $\sigma = \omega_1/\omega_2$. KAM showed that the tori that are *most easily destroyed* are those with *rational winding number*, often referred to as **resonant tori.** If you think for a minute you'll realize that those orbits are the *periodic* orbits! So, another surprise: the "nicest" orbits are destroyed first, but "almost all" orbits (those with *irrational* winding numbers) are preserved. This same concept is generalizable to higher dimensional tori.

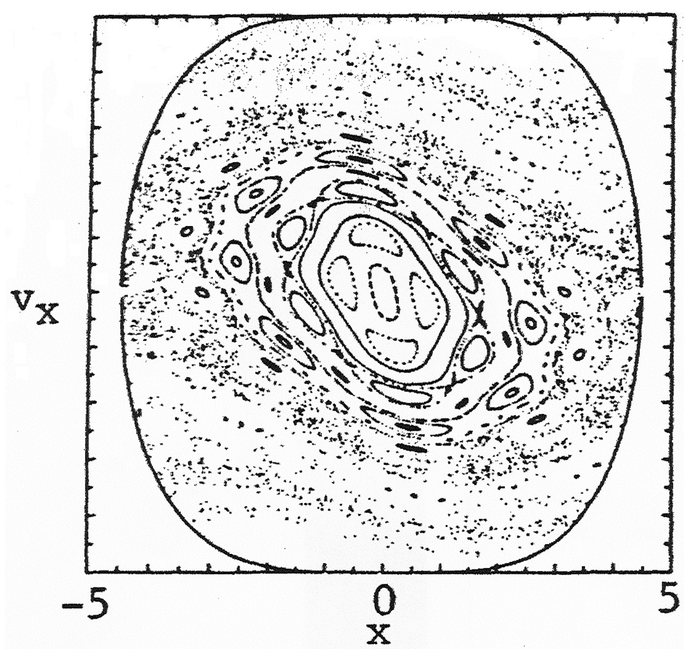
What happens when a resonant (rational) torus breaks up? The answer to this originated in a 1913 theorem by Birkhoff, originally suggested by Poincaré: it breaks up into a chain of *alternating elliptic and hyperbolic fixed points*! A suggestive diagrammatic view goes as follows: begin with a set of invariant tori, visualized as SOS plots (*i.e*. closed curves). When you turn on the perturbation, the rational tori are strongly deformed while the irrational ones are preserved (see figures below). Now, the intersection points between the original rational torus and the deformed torus are unchanged, *i.e.* fixed points, and if you calculate their nature you discover they alternate elliptic and hyperbolic, with the number of each equal to the integer denominator in the rational winding number: if $\sigma = \omega_1/\omega_2 = n_1/n_2$, where the *n*'s are integers, then there will be n_2 elliptic "islands" interspersed between n_2 unstable hyperbolic points where the chaotic motion originates.



Invariant tori (regular motion)

resonant torus distorts
(chaotic motion at "X" points)

The above figure shows an example for $n_2 = 2$, where the torus winding numbers σ_1 and σ_3 are irrational, while σ_2 is rational with $n_2 = 2$ resulting in two stable elliptical islands (marked with an "O") and two unstable hyperbolic points (marked with an "X"). Here's a nice example of a SOS plot from the neutral line system (a charged particle in a "neutral line" magnetic field -- see the homework). It displays this "KAM breakup" beautifully, as can be seen in the figure below:



In this figure you can clearly make out chains of "stable islands" with $n_2 = 4, 6, 8, 10$, and for some of those chains you can see the chains of unstable hyperbolic points and even chaos. I In

this system, increasing the nonlinearity could mean increasing the field ratio parameter, b_n, or a combined parameter depending on the conserved energy and linear momentum. This is just one of many interesting phase space structure plots possible with the neutral line system.

What happens if we increase the perturbation even further? The KAM theorem itself doesn't explicitly say, but other work does: after the rational winding number tori go chaotic, the irrational tori begin to break up also. As the perturbation grows, more and more irrational tori go unstable - can we tell which ones? Yes. I won't go into detail, but "ICBS" ("it can be shown") that, in the simplest cases, tori go unstable in order of their *degree of irrationality*. Without getting in to the math, suffice it to say that irrationality can be indeed be quantified and that "quadratic" irrationals (those that result from solutions of quadratic equations with integer coefficients) are the most irrational, so they are the last tori to go. The "most irrational" number is the *golden mean* $g = \frac{1}{2}\left(\sqrt{5}+1\right)$. Curiously, this number has held humankind's fascination since the ancient Greeks, and here it shows up again as the winding number of the most stable invariant torus.

By the way, whatever became of Poincaré's original question about the stability of the solar system? A full answer is still lacking, but the simpler case of three bodies in a plane can be answered, based on the KAM theorem and related results. If you consider the stability of the Earth's orbit under gravitational attraction from the Sun and Jupiter only, for example, full phase space would be 18 dimensional (6 for each body) and the motion of each would be represented by a torus embedded in this space (reduced in dimension to account for conservation of angular momentum and energy). Simplistically, in the unperturbed case ignoring the interaction between Earth and Jupiter, the ratio of winding numbers would be the same as the ratio of orbit periods (Jupiter period/Earth period), which happens to be about 11.862972, certainly not a small rational ratio and probably "fairly irrational" - hence we expect Earth's orbit to be "fairly stable" when the perturbation of Jupiter's gravitational field is "turned on". All this can be made more rigorous, but we will leave that to a course in advanced celestial mechanics.

Finally, KAM theory effectively tells us how systems go chaotic as we increase a nonlinear perturbation: it describes the "route to chaos" for Hamiltonian systems. The usual way you would increase a nonlinear perturbation would be to increase a parameter that controls the nonlinearity. For example, in the driven pendulum you could turn up the driving amplitude. Try it yourself: see what happens to the strobe plot if you run that system for smaller or large amplitudes -- you might be surprised at the results.

By the way, in the second half of the course, you'll learn that the many routes to chaos in dissipative systems can be quite different.

