Physics 220
Exam 2, Fall 2012

1) In a particular region of the planet Bovine, the gravitational acceleration near the surface is not constant as it is here, but in fact varies linearly with height so that the force of the Bovine gravity on an object of mass \( m \) as

\[
\vec{F}_g = -mg \frac{\gamma}{L} \hat{e}_y
\]

Here we have chosen the + \( y \)-direction to be up and the \( x \)-direction to be parallel to the ground. The constant \( L \) is the scale length of the variation in the gravity and \( g \) is the standard gravitation acceleration. In addition to gravity, objects moving through the Bovine atmosphere experience a linear drag force

\[
\vec{F}_{drag} = -c \vec{\dot{v}}
\]

Thus the net force on an object moving through the Bovine atmosphere is

\[
\vec{F}_{net} = \vec{F}_g + \vec{F}_{drag} = -c \vec{v}_x \hat{e}_x - c \vec{v}_y \hat{e}_y - mg \frac{\gamma}{L} \hat{e}_y
\]

The Black Angus Clan has developed a poo-apult that hurls a cow pie of mass \( m \) with an \textit{initial speed} \( v_0 \) and \textit{launch angle} \( \alpha \) relative to the horizontal. (Hint: Your life will be easier if you use the notation \( 2\gamma = c/m. \))

a) Is the net force on the cow pie as it flies through the air conservative? Explain. (3 points)
b) Write a differential equation that describes the motion of the pie in the \( y \)-direction. (3 points)
c) Write the solution for the \( y \)-position of the pie as a function of time assuming that it starts from \( x = 0 \) and \( y = 0 \) at \( t = 0 \). (Hint: You can just write the general form down and use initial conditions to solve for the constants) (7 points)
d) At what time after launch, does the cow pie first hit the ground? (3 points)
e) Write and solve a differential equation that describes the velocity of the pie in the \( x \)-direction as a function of time again assuming that it starts from \( x = 0 \) and \( y = 0 \) at \( t = 0 \). (7 points)
f) Using the result of part d) find the position of the particle as a function of time. (7 points)
g) Determine the Range of the pie as a function of the launch angle. (3 points)

2) A Holstein in a nearby region of space is observed to be moving under the influence of a central force and following a trajectory that is given by the equation

\[
r = \frac{r_0}{1 + \theta}
\]

where \( r_0 \) is a positive constant.

a) Find the central force that causes the Holstein to follow this path (Remember this must be \( f(r) \). I don’t want to see any \( \theta \)'s showing up in your final answer.) (9 points)
b) Circular orbits are possible in any central force. Are the “nearly” circular orbits in the force law calculated in part a) stable or unstable? How do you know? (6 points)
c) Use conservation of angular momentum to calculate \( \theta \) as a function of time. Assume that \( \theta = 0 \) when \( t = 0 \). (6 points)
d) Find \( r \) as a function of time. (6 points)
e) How long does it take the Holstein to reach the center of the circle, i.e \( r=\theta? \) (6 points)
3) Elsewhere on the planet Bovine, the gravitational acceleration near at the surface is simply a constant $g = 9.8 \text{ m/s}^2$ in the downward direction (just like Earth). The Hereford’s there are conducting detailed experiments concerning motion in rotating reference frames. In one particular experiment a small calf (who has a mass of 25kg) walks radially inward with a speed of 2 m/s on a merry-go-round rotating with a constant angular velocity $\omega = 0.5 \text{ rad/s}$. Choose the origin of both the fixed and rotating frames to be the center of the merry-go-round. (1 point free)

a) What is the **magnitude and direction** of the *coriolis* force when the calf is 5 m from the center? Clearly indicate this force on a diagram that includes the coordinates used and the direction of rotation. (8 points)

b) What is the magnitude and direction of the *centrifugal* force when the calf is 5 m from the center. Again, clearly indicate this force on a diagram that includes the coordinates used and the direction of rotation. (8 points)

c) What is the minimum value of the coefficient of friction if the calf is not slipping at this point. (8 points)

d) Still assuming that the calf is 5 m from the center and that $\omega = 0.5 \text{ rad/s}$, in what direction and with what speed should the calf walk if the coriolis and the centrifugal forces are to exactly cancel each other out? (8 points)
1) Since there is dissipation, this is NOT a conservative force.

b) \[ m \ddot{y} + c \dot{y} + kx = 0 \]
\[ 2 \gamma = \frac{c}{m} \]
\[ \omega^2 = \frac{k}{m} \]
\[ \ddot{y} + 2 \gamma \dot{y} + \omega^2 x = 0 \]

2) solve:
\[ y(t) = e^{-\gamma t} \left[ A \cos(\omega t) + B \sin(\omega t) \right] \]
\[ \omega^2 = \omega^2 - \delta^2 \]
\[ a) \quad \text{at } t = 0, \quad y = 0 = A \Rightarrow y(t) = B e^{-\gamma t} \sin(\omega t) \]
\[ \dot{y} = B \left[ -\delta e^{-\gamma t} \sin(\omega t) + e^{-\gamma t} \cos(\omega t) \omega \right] \]
\[ b) \quad \text{at } t = 0 \]
\[ y(0) = V_0 \sin \delta = B \left[ \omega \right] \Rightarrow B = \frac{V_0 \sin \delta}{\omega} \]

3) Hit ground when
\[ t = \frac{\pi}{\omega} \]

4) \[ m \frac{d^2 x}{dt^2} = -c V_x \quad \Rightarrow \quad \frac{dV_x}{dt} = -\frac{c}{m} V_x \]
\[ \int \frac{dV_x}{V_x} = -2 \gamma \int_0^t dt \quad \Rightarrow \quad \ln \left( \frac{V}{V_0 \cos \delta} \right) = -2 \gamma t \]
\[ V = V_0 \cos \delta \quad e^{-2 \gamma t} \]

5) \[ \int \frac{dx}{dt} = V_0 \cos \delta \int e^{-2 \gamma t} \, dt \]
\[ x = V_0 \cos \delta \left( -\frac{1}{2 \gamma} e^{-2 \gamma t} \right) \bigg|_0^t = \left[ \frac{V_0}{2 \gamma} \cos \delta \left( 1 - e^{-2 \gamma t} \right) \right] = x(t) \]

6) \[ R = x(t) = \frac{V_0}{2 \gamma} \cos \delta \left( 1 - e^{-\frac{2 \gamma t}{\omega}} \right) \]
2a) \( r = \frac{r_0}{1+\theta} \Rightarrow u = \frac{1}{r} = \frac{1}{r_0} (1+\theta) \); \( \frac{du}{d\theta} = \frac{1}{r_0} \); \( \frac{d^2u}{d\theta^2} = 0 \)

\[
\frac{d^2u}{d\theta^2} + u = -\frac{1}{m^2 u^2} f(u)
\]

\[
U = -\frac{1}{m^2 u^2} f(u)
\]

\[
f(u) = -m^2 u^3 \Rightarrow \frac{f(r)}{r^3} = -\frac{m^2}{r^3}
\]

b) Not stable; \( n > 3 \) to be stable, thus \( n = -3 \)

c) \( r^2 \frac{d\theta}{dt} = l \)

\[
\frac{r_0^2}{(1+\theta)^2} \frac{d\theta}{dt} = l \Rightarrow \int_0^\theta \frac{d\theta}{(1+\theta)^2} = \int_0^t \frac{dt}{r_0^2}
\]

\[
-\frac{1}{1+\theta} \bigg|_0^\theta = -\frac{1}{1+\theta} + \frac{1}{1} = \frac{\theta}{r_0^2}
\]

\[
\frac{1}{1+\theta} - 1 = -\frac{\theta}{r_0^2} \Rightarrow \frac{\theta}{r_0^2} = \frac{1}{1+\theta} - 1
\]

\[
\theta = \frac{1}{1-\frac{1}{r_0^2} t} - 1
\]

\[
\frac{1}{1+\theta} = 1 - \frac{\theta}{r_0^2} t \Rightarrow it = \frac{1}{1-(\theta^2 t)}
\]

\[
\theta = \frac{1}{1-\frac{1}{r_0^2} t} - 1 = (1-\frac{1}{r_0^2} t)^{-1} - 1
\]

d) \( r = \frac{r_0}{1+\theta} = \frac{r_0}{(1-\frac{1}{r_0^2} t)^{-1}} = \frac{r_0}{(1-\frac{1}{r_0^2} t) - 1} = \frac{r_0 (1-\frac{1}{r_0^2} t)}{1-\frac{1}{r_0^2} t} = r
\)

e) Reaches center \( (r=0) \) when \( (1-\frac{1}{r_0^2} t) = 0 \)

\[
\Rightarrow \left[ t = \frac{r_0^2}{2} \right]
\]
\[ \omega = 0.5 \text{ s}^{-1} \hat{e}_z \quad m = 25 \text{ kg} \]
\[ v' = 5 \text{ m/s} \hat{e}_x \quad v' = -2 \text{ m/s} \hat{e}_y \]

a) \[ F_{cor} = -2m (\omega \times v') = -2(25 \text{ kg}) (0.5 \text{ s}^{-1} \hat{e}_z) \times (-2 \hat{e}_y) \]
\[ = 50 \left( \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \right) (\hat{e}_z \times \hat{e}_x) = 50 \text{ N} \hat{e}_y = F_{cor} \]

b) \[ F_{cent} = -m \left[ \omega \times (\omega \times v') \right] = -(25 \text{ kg}) \left[ (0.5 \text{ s}^{-1}) \hat{e}_z \times (0.5 \hat{e}_z \times 5 \hat{e}_y) \right] \]
\[ = -31.25 \left( \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \right) (\hat{e}_z \times \hat{e}_y) = 31.25 \text{ N} \hat{e}_x = F_{cent} \]

c) \[ |F_{reaction}| = mg = \sqrt{F_{cor}^2 + F_{cent}^2} \]
\[ \Rightarrow M = \left( \frac{1}{mg} \right) \sqrt{F_{cor}^2 + F_{cent}^2} \]
\[ = \frac{1}{25(9.8)} \sqrt{(50)^2 + (31.25)^2} = 0.241 \]

d) \[ F_{cor} = -31.25 \hat{e}_x \]. For this to be true, the sled must be walking in the \(-\hat{e}_y\) dir.

With a speed \[ v = \frac{1}{2} \omega r = (0.5)(0.5)(5) = 1.25 \text{ m/s} \]

Double check
\[ F_{cor} = -2(25) (0.5 \hat{e}_z \times -1.25 \hat{e}_y) = -31.25 \text{ N} \hat{e}_x \]